Magnetohydrodynamic Micromixing

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ABSTRACT

Modeling of magnetohydrodynamic (MHD) pumping in an annular microchannel is reduced to a two-dimensional Poisson equation for the azimuthal velocity. A method for enhancing the mixing of fluids is proposed, and is proven to yield linear growth in time of the fluid interface in the absence of molecular diffusion.

Keywords: MHD, microfluids, mixing

1 INTRODUCTION

Efficient mixing of fluids in the low Reynolds number flows typical of micro total analysis systems (μTAS) and chemical microreactor technologies requires novel approaches. The lack of turbulence and the fabrication and reliability challenges of incorporating moving stirrers into microdevices has inspired investigation of other mixing mechanisms. Magnetohydrodynamic forces provide a pumping mechanism in conductive liquids such as biological fluids, and have recently been exploited in the design of suitable electrode patterns for micromixing devices [1].

In this paper the effects of an alternating current MHD force upon fluid in a cylindrically symmetric channel are examined. Suitable approximations reduce the full MHD equations to a Poisson equation in the channel cross-section. Solutions of this model accurately match experimental observations of fluid circulation in an annular channel [2]. Analytical solutions for the fluid velocity are zero at the walls (no-slip boundaries) and have a maximum near the centre of the channel. As a result of this velocity profile, fluids initially separated into semicircular portions of the annular channel are forced by the MHD pumping to pass through each other. The length of the interface between the fluids is described by a simple approximation, and is shown to grow linearly in time, thus enhancing mixing. No special patterning of electrodes [1] is necessary, and the simplicity of this approach makes it attractive as a general-purpose micromixing mechanism.

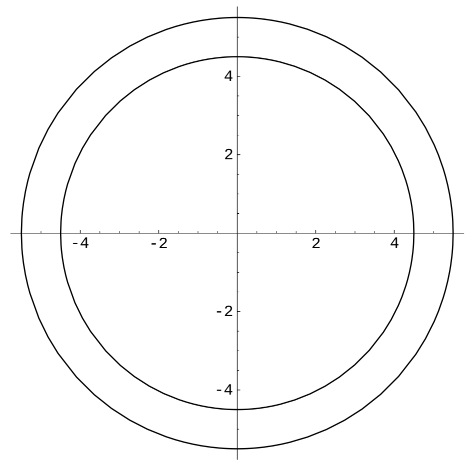

Figure 1: Top view of channel: inner radius $a = 4.5$ mm, outer radius $b = 5.5$ mm. Electrodes line the inner and outer walls. The electromagnet lies underneath the circular channel, so magnetic field lines point out of the page.

2 MHD PUMPING

Magnetohydrodynamic actuation is potentially useful as a pumping mechanism for ionic fluids in closed microchannels. Cyclical chemical reactions are natural candidates for experiments in annular microchannels. Recent work [2] has utilised circular micromachined channels of rectangular cross-section to demonstrate the feasibility of a continuous flow polymerase chain reaction (PCR) for DNA amplification. An electromagnet is placed below the center of the annulus, so the magnetic field is vertically out of the page in Figure 1. The curved inner and outer walls of the annulus are plated and act as electrodes for a current which is in phase with the high frequency alternating magnetic field. As a result of the orthogonal magnetic and electric fields, the fluid experiences a Lorentz force which pumps it around the annulus. This work aims to accurately model this phenomenon, and in section 3 we exlore the micromixing capability of the MHD actuation.

Lemoff and Lee [3] introduced a model of MHD flow based upon Poiseuille flow in a straight channel and showed reasonable comparison with experimental results. In this section we derive an analogous model for circular channels. The equations of incompressible MHD are [4]:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\partial \mathbf{B}/\partial t, && (1)\\ \nabla \times \mathbf{B} &= \mu \mathbf{j}, && (2)\\ \nabla \cdot \mathbf{B} &= 0, && (3)\\ \nabla \cdot \mathbf{j} &= 0, && (4)\\ \nabla \cdot \mathbf{v} &= 0, && (5)\\ \mathbf{j} &= \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), && (6)\\ \rho D\mathbf{v}/Dt + \nabla p &= \mathbf{j} \times \mathbf{B} + \eta \nabla^2 \mathbf{v}, && (7) \end{aligned}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{j}$ is the current density, $\mathbf{v}$ is the velocity of conductors, μ is the permeability of free space, σ is electrical conductivity and ρ, p and η are, respectively, the density, pressure and (absolute) viscosity of the fluid. The total derivative $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is used in equation (7). We follow Lemoff and Lee in assuming that the main MHD effects are represented by the Lorentz force on the fluid, i.e. we assume that the effects of the $\mathbf{v} \times \mathbf{B}$ term in equation (6) is negligible, and check this *a posteriori*. The equation of interest is then (7) which is to be solved for the fluid velocity, given the imposed current density $\mathbf{j}$ and the magnetic field $\mathbf{B}$.

The experimental geometry is as in Figure 1. The channel cross-section is rectangular, with width 1 mm and depth 0.5 mm. The electrodes cover the entire inner and outer walls of the channel, and pass an AC current of average amplitude $I/\sqrt{2}$ through the fluid. Assuming cylindrical symmetry and a channel depth h, this gives an average current density vector

$$\mathbf{j} = \frac{I}{2\sqrt{2}\pi h r} \mathbf{e}_r, \qquad (8)$$

where $\mathbf{e}_r$ is the unit vector in the radial direction. The imposed magnetic field is taken to point in the vertical z-direction (out of the page in Figure 1) and is written as

$$\mathbf{B} = B/\sqrt{2}\mathbf{e}_z, \qquad (9)$$

for AC-averaged amplitude B. The averaged vector product $\mathbf{j} \times \mathbf{B}$ appearing on the right hand side of equation (7) can then be evaluated to give the Lorentz force

$$\mathbf{j} \times \mathbf{B} = -\frac{BI}{4\pi h r} \mathbf{e}_\theta, \qquad (10)$$

where $\mathbf{e}_\theta$ is the unit vector in the azimuthal direction.

Equation (7) admits a cylindrically symmetric steady solution with radial velocity and axial velocity both zero, and azimuthal velocity and pressure depending only on r and z:

$$\begin{aligned} v_r &= 0, && (11)\\ v_z &= 0, && (12)\\ v_\theta &= v(r,z), && (13)\\ p &= p(r,z). && (14) \end{aligned}$$

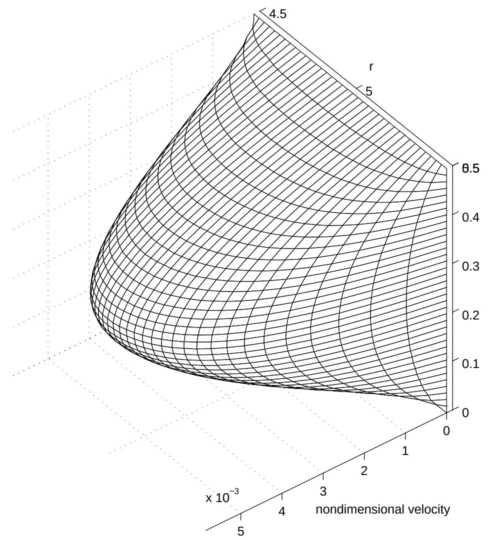

Figure 2: Finite element solution for velocity profile. Depth is measured along z axis, with channel bottom at $z = 0$, top at $z = h = 0.5$ mm. The channel width is 1 mm, with $r = 4.5$ corresponding to the inner wall, $r = 5.5$ to the outer.

Substituting these and (10) into (7) yields a Poisson equation for the steady azimuthal velocity:

$$\nabla^2 v - \frac{v}{r^2} = -\frac{BI}{4\pi r h \eta}. \qquad (15)$$

This may be solved using a Fourier-Bessel expansion, or by straightforward numerical computation using a two-dimensional finite element package, such as Matlab's PdeTool. The boundary conditions at the sides ($r = a$ and $r = b$), top ($z = h$), and bottom ($z = 0$) of the channel are no-slip:

$$\begin{aligned} v(a,z) &= 0, && (16)\\ v(b,z) &= 0, && (17)\\ v(r,0) &= 0, && (18)\\ v(r,h) &= 0. && (19) \end{aligned}$$

Note the resulting flow has circular streamlines, and is an exact solution of the (fully nonlinear) Navier-Stokes equation (7). We stress that no assumption on the Reynolds number of the flow has been made—all that is required is the absence of turbulent instabilities, as is commonly the case in microfluid flows.

We present results for a channel of inner radius $a =$ 4.5 mm, outer radius $b = 5.5$ mm, and depth $h = 0.5$ mm. The current amplitude is $I = 32$ mA, and the magnetic field is $B = 7$ mT, with the fluid viscosity taken to be that of water at 15 degrees: $\eta = 0.001$ kg/m/s. Plotted in Figure 2 is the quantity

$$-v(r,z) \times \left(\frac{BI}{4\pi h \eta}\right)^{-1}$$

evaluated at each (r,z) point in the channel cross-section. This has a maximum near the middle of the channel, with value 0.0057, corresponding to a dimensional velocity of magnitude

$$v = 0.0057\frac{BI}{4\pi h\eta} \tag{20}$$
$$= 0.0057\frac{7\times 32}{4\pi\times 0.5\times 1} \tag{21}$$
$$= 0.20 \text{ mm/s}. \tag{22}$$

Note that the average of velocity over the whole cross-section is half of the maximum velocity, i.e., 0.1 mm/s. In experiments [2] tracer particles are found to move with speeds that cluster around and above the average, but below the maximum values predicted from the model. Although the dimensions of the micromachined device described above are relatively large, current work on scaling down of the geometry using rapid prototyping of SU-8 also indicates agreement of the model with experiment.

3 MIXING APPLICATION

The maximum of the fluid velocity near the center of the channel, and the no-slip condition at the walls raise the possibility of utilising the annular geometry as a micromixing device. To simplify the discussion we consider a two-dimensional channel, i.e., we neglect all dependence on the vertical coordinate z. The azimuthal velocity profile is then a function only of the radius r—this is a good approximation to the full three-dimensional profile if the channel has a high depth-to-width ratio (except in boundary layers near the top and bottom). When $v = v(r)$, equation (15) reduces to the ordinary differential equation

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = -\frac{BI}{4\pi r h\eta}, \tag{23}$$

with no-slip conditions $v(a) = v(b) = 0$. Setting $\alpha = -BI/4\pi h\eta$ for convenience, we can write the solution of (23) as

$$v(r) = \frac{\alpha}{2(a^2-b^2)r}\left(a^2b^2\ln\frac{a}{b} + a^2r^2\ln\frac{r}{a} + b^2r^2\ln\frac{b}{r}\right). \tag{24}$$

This profile contains corrections to the Poiseuille profile for a straight channel (which is quadratic in r) due to the curvature effects. Indeed, by defining the radius of curvature of the channel $R = (a+b)/2$ and the half-width $w = (b-a)/2$, it can be shown that (24) reduces to the Poiseuille form

$$v(r) \sim \frac{-\alpha}{a+b}(r-a)(b-r) \tag{25}$$

in the limit $w/R \ll 1$. Since the ratio w/R may be written as

$$\frac{w}{R} = \frac{b-a}{b+a} = \frac{1-\frac{a}{b}}{1+\frac{a}{b}},$$

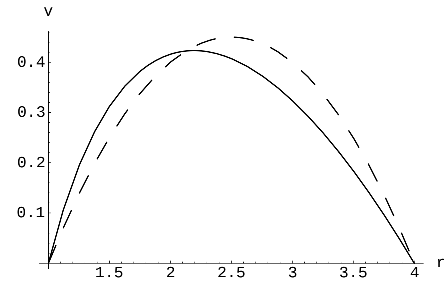

Figure 3: Exact velocity profile in a curved channel (24) (solid), and the approximation (25) (dashed). Here the outer radius b is 4 times the inner radius a.

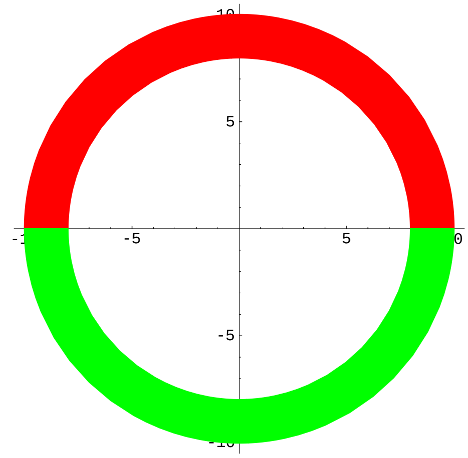

Figure 4: Initially separated fluids.

it is always less than unity, and so (25) is a good approximation to (24) except for very wide channels. In Figure 3 we plot (24) and (25) for $a/b = 0.25$.

The micromixing proposal is based upon the observation that the fluid velocity is higher in the center of the channel than at the sides. Therefore two initially separated fluids as in Figure 4 are forced to pass through each other (Figure 5). The resulting increase in interface length then enhances mixing by molecular diffusion. To quantify the increase in interface length, we define the angular position $\theta(r)$ of an element of the interface which is initially along the radius $\theta = 0$. At a later time t this element has traveled along the circular streamline to an angular position (assuming zero molecular diffusion)

$$\theta(r,t) = v(r)t \bmod 2\pi.$$

This is the equation used to generate Figure 5. The length of the interface is found by the usual method for calculating distance along a curve:

$$l(t) = \int_a^b \sqrt{1+(\theta'(r))^2}dr$$
$$= \int_a^b \sqrt{1+t^2\,(v'(r))^2}dr,$$

and the integral may be calculated in closed form using (25). The long-time asymptotics are given by

$$l(t) \sim \frac{-\alpha(b-a)^2}{2(a+b)} t + o(t) \text{ as } t \to \infty.$$

This linear growth in time enhances mixing and the simplicity of the design makes it attractive as a general-purpose micromixing mechanism. Further work is needed to clarify the dispersive effects of molecular diffusion which has been assumed to be zero in this discussion

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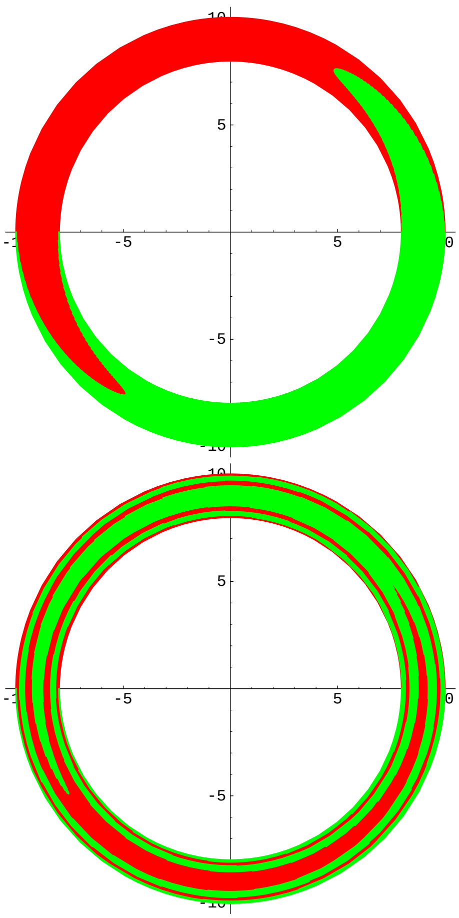

Figure 5: Growth of interface at later times.