Nonlinear Analytical Reduced-Order Modeling of MEMS

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ABSTRACT

We present a method for constructing a nonlinear, analytical, reduced-order model for MEMS-structures. The method is based on the center manifold approach for describing nonlinear modes of mechanical systems, in combination with techniques for predicting eigenvalues and eigenvectors. When considering a given mode of the mechanical system, this method, unlike modal truncation procedures, takes into account the effects of nonlinearities of the non-modeled modes. Here we present the details of the method and demonstrate its capabilities by applying it to an accelerometer-structure.

Keywords: Reduced-order models, nonlinear normal modes, MEMS.

1 INTRODUCTION

The emergence of a MEMS industry has led to a great increase in the need for analyzing systems with coupled energy domains. In MEMS structures, we often encounter coupled mechanical/electrical or mechanical/thermal problems. Typically, significant disparities in temporal and spatial scales exist between the domains, leading to complicated and computationally expensive simulations. To overcome such problems, it is a common procedure to select a primary domain for describing the system and then include the effects of other domains in terms of equivalent modules recognized by the primary domain. An example is the use of the electronic circuit simulator SPICE for describing mechanical/electrical systems, representing the mechanical parts by electrical equivalent circuits [1].

The importance of normal modes in linear dynamical systems is indisputable. They are used to decouple the governing equations of motions, and to evaluate the system response. Furthermore, linear superpositioning can be used in cases where nonlinearities are insignificant. For MEMS, truncated low-order models can be established this way, using a summation over only selected, important modes [2]. In the presence of significant nonlinearities, which often is the case for MEMS, the simple truncated models tend to be too imprecise. However, by performing additional simulations of the complete system, the technique can be enhanced, whereby nonlinear effects in the truncated mode set can be obtained [3]. Normally, this may lead to a large number of simulations.

As an alternative, we may use a procedure where both nonlinearities and, indirectly, a wider modal space are included, based on the creation of nonlinear, reduced-order models [4]. This is done by focusing on the inherent nonlinear behavior from the outset of the analysis, using nonlinear normal mode (NNM) theory [5].

While the above formalisms were developed primarily for numerical simulations, we will introduce a procedure that makes it possible to create parameterized macro-models suitable for use in predictive and transparent analysis and design.

2 NONLINEAR NORMAL MODES

Vakakis extended the definition of normal modes of classical vibration theory to the nonlinear case [6]. He defined a nonlinear normal mode (NNM) in an undamped, discrete or continuous system as a synchronous, periodic oscillation where all material points of the system simultaneously reach their extreme values or pass through zero. Clearly, when a discrete system vibrates in an NNM, the corresponding oscillation can be represented in the configuration space as a modal line (deflection path), as indicated in Figure 1. Linear systems possess linear modal lines, while nonlinear systems generally have nonlinear modal lines.

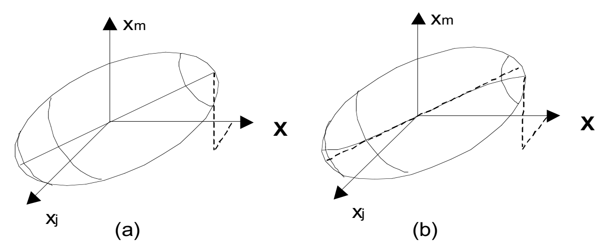

Figure 1: Linear (a) and nonlinear (b) modal lines in the configuration space. **X** indicates the excitation of the mode.

2.1 Center manifold theory

Shaw and Pierre [7] first applied center manifold theory (CMT) to nonlinear modal analysis. In CMT, modal motion is confined to curved manifolds within the system's phase space. Later, Pesheck and Pierre [4] developed a rigorous methodology for reducing a large, nonlinear system of equations to a more manageable subset. In this subset, the configuration space is confined to a high-dimensional curved manifold, parameterized by the retained degrees of freedom of the subset. However, this technique is strictly valid only for systems with low damping. Here follows a brief outline of the formulation.

Consider a dynamical system of N degrees of freedom, expressed in the following form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{\mathbf{nl}}(\mathbf{x}) = \mathbf{0} \tag{1}$$

where **M** is a mass matrix, **K** is the linear stiffness matrix, and $\mathbf{f}_{\mathbf{nl}}$ represents the nonlinearities of the stiffness. We assume that the nonlinearities can be expressed by means of quadratic and cubic terms. Using linear modal coordinates **q**, we may rewrite eq. (1) as

$$\mathbf{I}\ddot{\mathbf{q}} + \hat{\mathbf{e}}\mathbf{q} + \acute{\mathbf{a}}\mathbf{q}^{2x} + \hat{\mathbf{a}}\mathbf{q}^{3x} = 0 \tag{2}$$

where **I** is the identity matrix, _ is the new stiffness matrix, $\acute{\mathbf{a}}$ and $\hat{\mathbf{a}}$ are the second- and third-order stiffness matrices, respectively, and $\mathbf{q}^{2x}$ and $\mathbf{q}^{3x}$ contain all second and third order combinations of the modal coordinates, respectively. Eq. (2) may be recast into the following general form:

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}, \mathbf{q}) \end{cases} \tag{3}$$

where $\mathbf{q}^T = [q_1\ q_2 \ldots q_N]^T$ and $\mathbf{p}^T = [p_1\ p_2 \ldots p_N]^T$ represent the generalized positions and velocities, while $\mathbf{f}^T = [f_1\ f_2 \ldots\ f_N]^T$ represents the position and velocity dependent forces. Since we are searching for a multi-mode manifold corresponding to a subset S_M of modes, we may express the coordinates as:

$$\begin{aligned} &\begin{cases} q_k = u_k \\ p_k = v_k \end{cases} \quad \text{for } k \in S_M \\ &\begin{cases} q_j = X_j(\mathbf{u}, \mathbf{v}) \\ p_j = Y_j(\mathbf{u}, \mathbf{v}) \end{cases} \quad \text{for } j \notin S_M \end{aligned} \tag{4}$$

Substituting this into the governing equations, we obtain a set of equations with no readily available solutions. However, we write an approximate solution for the coordinates X_j and Y_j ($j \notin S_M$) in the form of a polynomial expansion in the coordinates u_k and v_k ($k \in S_M$) as follows:

$$X_j = \begin{bmatrix} \sum\limits_{k \in S_M} a_{1,j}^{k} u_k + a_{2,j}^{k} v_k \\ + \sum\limits_{k \in S_M} \sum\limits_{l \in S_M} a_{3,j}^{k,l} u_k u_l + a_{4,j}^{k,l} u_k v_l + a_{5,j}^{k,l} v_k v_l \\ + \sum\limits_{k \in S_M} \sum\limits_{l \in S_M} \sum\limits_{q \in S_M} a_{6,j}^{k,l,q} u_k u_l u_q + a_{7,j}^{k,l,q} u_k u_l v_q \\ + a_{9,j}^{k,l,q} u_k v_l v_q + a_{9,j}^{k,l,q} v_k v_l v_q + \cdots \end{bmatrix} \tag{5.a}$$

$$Y_j = \begin{bmatrix} \sum\limits_{k \in S_M} b_{1,j}^{k} u_k + b_{2,j}^{k} v_k \\ + \sum\limits_{k \in S_M} \sum\limits_{l \in S_M} b_{3,j}^{k,l} u_k u_l + b_{4,j}^{k,l} u_k v_l + b_{5,j}^{k,l} v_k v_l \\ + \sum\limits_{k \in S_M} \sum\limits_{l \in S_M} \sum\limits_{q \in S_M} b_{6,j}^{k,l,q} u_k u_l u_q + b_{7,j}^{k,l,q} u_k u_l v_q \\ + b_{9,j}^{k,l,q} u_k v_l v_q + b_{9,j}^{k,l,q} v_k v_l v_q + \cdots \end{bmatrix} \tag{5.b}$$

Inserting this approximation in (2), we obtain the coefficients (a's and b's) in (5). Although not explicitly stated here, the solution may be given in an analytical form [4]. Generally, these coefficients can be expressed through the following coupled equations:

$$\begin{aligned} \mathbf{D}(\grave{\mathbf{u}})\mathbf{a}^{\mathbf{xx}} &= \mathbf{h}_1(\acute{\mathbf{a}}) \\ \mathbf{E}(\grave{\mathbf{u}})\mathbf{a}^{xxx} &= \mathbf{h}_2(\acute{\mathbf{a}}, \hat{\mathbf{a}}, \mathbf{a}^{\mathbf{xx}}) \end{aligned} \tag{6}$$

where the matrices **D** and **E** are known functions of the linear eigenfrequencies, the vectors $\mathbf{a}^{xx}$ and $\mathbf{a}^{xxx}$ represent the coefficients in (5). $\mathbf{a}^{xx}$ is obtained from the first of these equations where $\mathbf{h}_1$ is a known function of the second order stiffness matrix. Using this result, $\mathbf{a}^{xxx}$ is obtained from the second equation where $\mathbf{h}_2$ is a function of $\mathbf{a}^{xx}$ and the second and third order stiffness matrices.

Hence, using the solution for the coordinates X_j and Y_j ($j \notin S_M$) in the non-linear terms of (2), we obtain a manageable formulation of the reduced-order, nonlinear problem expressed in terms of the coordinates for the modes belonging to S_M.

3 DESIGN-PARAMETER SENSITIVITY

From the above general formalism, we now proceed to develop a parameterized model formalism suitable for predictive design. One possibility is to sample the design space of interest to obtain a suitable parameter set. However, this may require extensive simulations. Instead, we adopt a strategy based on Taylor-series expansions of the above formalisms with respect to the design parameters.

First, our problem is normally stated in terms of general coordinates as in (1), instead of the modal coordinates as in (2). We therefore generate an approximate formulation in the form of (2) by a linear expansion of the eigenvectors of the original system in terms of the design parameters. Various methods exist for finding these derivatives, such as Nelson's method [8], finite-difference methods, modal methods, and modified modal methods, although computational cost and accuracy may vary [9]. For problems with repeated (degenerate) eigenvalues and their associated eigenvectors, special methods must be applied [10][11].

Having calculated the derivatives by means of one of the above-mentioned methods, we may write:

$$\widetilde{\mathbf{Q}} = \mathbf{Q}_0 + \left(\frac{\partial}{\partial g} \mathbf{Q}_0 \right) \Delta g \qquad (7)$$

where $\mathbf{Q}_0 = [\mathbf{q}_1\ \mathbf{q}_2\ \ldots\ \mathbf{q}_N]$ is the initial eigenvector matrix, $\widetilde{\mathbf{Q}}$ is the perturbed eigenvector matrix, and g is a design-parameter (typically length, width, etc). Assuming that the dependence of the stiffness matrices on the design parameters are known, we may proceed to rewrite (2) in the following form:

$$\mathbf{I}\ddot{\widetilde{\mathbf{q}}} + \widetilde{\mathbf{K}}(g)\widetilde{\mathbf{q}} + \widetilde{\mathbf{a}}(g)\widetilde{\mathbf{q}}^{2x} + \widetilde{\hat{\mathbf{a}}}(g)\widetilde{\mathbf{q}}^{3x} = \mathbf{0} \qquad (8)$$

However, we also need an approximate solution of (6), which involves the following derivatives:

$$\frac{\partial}{\partial g}\mathbf{a}^{xx} = \mathbf{D}^{-1}\left(\frac{\partial}{\partial}\mathbf{h}_1 - \left(\frac{\partial}{\partial g}\mathbf{D} \right)\mathbf{a}^{xx} \right) \qquad (9)$$

and the corresponding ones for $\mathbf{a}^{xxx}$.

Hence we have constructed approximate solutions both for the stiffness matrices and for the coefficients in the polynomial expansion. Inserting the approximate solutions of the non-modeled modes into equation (8), we have obtained a reduced, parametrized, nonlinear model of the system.

4 SIMULATION RESULTS

The above method was implemented in the commercial mathematical program Mathcad and tested on an industrialized dual-axis accelerometer[1] as shown in Fig. 2.

The accelerometer was realized in single-crystalline silicon by a bulk micro-machining process, where the z-axis acceleration is measured by piezoresistors implanted in the inner (light gray) beams, while the x-axis acceleration is detected by piezoresistors implanted in one of the outer (light gray) beams (torsional deformation).

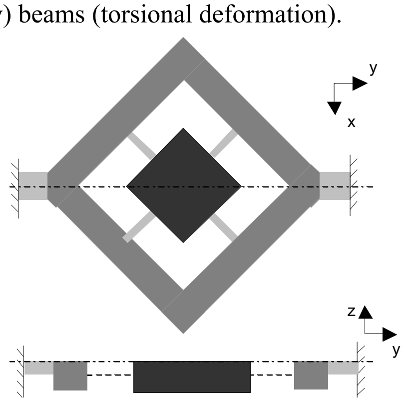

Figure 2: Schematic view of the dual-axis accelerometer investigated. Detection is done for x- and z-axis acceleration (Courtesy of SensoNor ASA).

[1] Patent pending EP 00305807.0, SensoNor ASA.

A preliminary analysis indicates that there are two dominant modes in the system, the z-axis deflection of the central mass (Mode 2) and the torsion of the assembly about the y-axis (Mode 1). Hence, these modes are chosen as our subset S_M. In general, the modes to be included in the subset should be selected with care.

Here we primarily investigate the validity of the design parameter sensitive solution of Section 3 by comparisons with the complete reduced-order formalism of Section 2. In the former, approximate results are obtained by liner expansion in terms of a given design parameter from a known starting point, while in the latter, the results were always calculated using the complete reduced model. The design parameter considered is the length L of the left-side outer beam.

In Fig. 3 we show the change in the small-signal eigenvalues of the two dominant modes with increasing L. Although this result does not depend on the nonlinearities in the stiffness matrix, it illustrates the overall sensitivity of the system to the parameter chosen, and how well this is reproduced using the linear expansion of Section 3. We notice that the linear expansion provides a good indication of the sensitivity of the eigenfunctions to the design parameter L, and reproduces the results from the complete reduced-order model quite well for up to 10 – 20 percent increase in L.

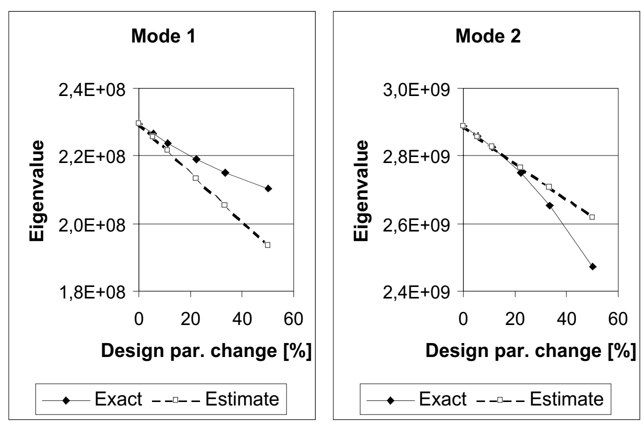

Figure 3: Eigenvalues of the two dominant modes versus the relative increase in the design parameter L.

To illustrate the effects of the stiffness nonlinearties, we show in Fig. 4 a similar comparison of the sensitivity of a dominant second order term in the stiffness matrix. We again observe, just as for the eigenvalues, that the series expansion reproduces the results from the complete reduced-order model quite well for up to 10 – 20 percent increase in L.

As a further illustration of the nonlinear effects, we have considered the coupling of Modes 1 and 2. In Fig. 5, we have plotted the temporal variation in the excitation of Mode 1 resulting from an excitation of the system in Mode 2. The figure shows the excitation in Mode 1 calculated from the complete reduced-order model for a nominal value of the design parameter L and for an 11 percent increase in L. In addition, the latter was calculated using the linear expansion procedure. Again, we observe that the linear expansion provides a good representation of the excitation predicted by the complete model.

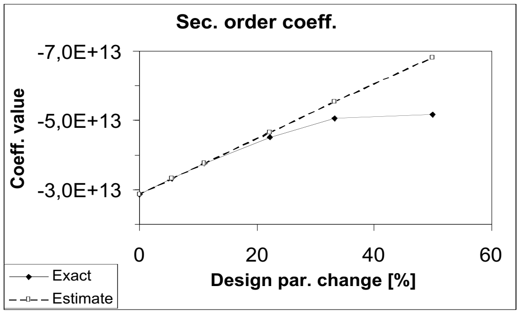

Figure 4: Dependence of a dominant second order stiffness coefficient on the relative change of the design parameter L, calculated with the complete reduced-order model and its linearization.

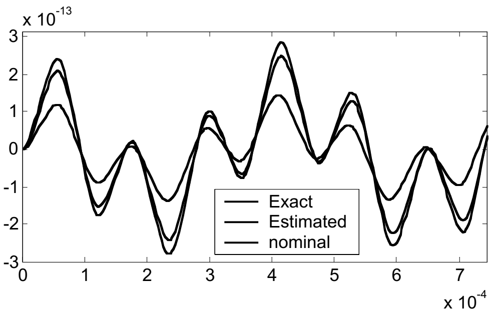

Figure 5: Comparison of mode excitation caused by nonlinear mode coupling, calculated with the complete reduced-order model (solid line) and its linearization (dotted line) at 11 percent change in the design parameter L. Also shown: the excitation at the nominal value of L.

5 DISCUSSION AND SUMMARY

In this paper, we have shown how a reduced-order description of nonlinear, dynamical MEMS systems can be formulated analytically, allowing it to be used in predictive device design. A version of the formalism linearized with respect to design parameters is shown to be quite suitable for sensitivity analysis and design optimization.

Typically, multi-energy domain problems, such as electrical/mechanical domain interaction, arise in MEMS structures, which easily lead to complicated simulations. A possible solution is to select a primary domain to describe the system and then include the effects of other domains in terms of equivalent modules recognized by the primary domain.

SPICE is a widely recognized circuit simulator for use in the electrical domain. This simulator requires the circuit elements to be described by analytical expressions. Therefore, using suitable equivalent descriptions of the mechanical elements of a mixed electrical/mechanical system, SPICE lends itself very well for predictive analysis of such systems. The reduced-order models developed here are examples of analytical descriptions that are adaptable to the SPICE environment. Another example is the interaction of an electrically driven vibrating plate with a squeezed gas film [12]. The next, logical step is therefore to implement these types of models in SPICE.

6 ACKNOWLEDGEMENT

This work is supported by a grant from the Research Council of Norway, and support from SensoNor ASA.

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